

Grothendieck Fibrations

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Definition

Preliminary definitions

In this section, we have two categories $\mathcal{B}$ and $\mathcal{E}$, and a functor $p : \mathcal{E} \rightarrow \mathcal{B}$.

Definition 1 (Refinement)

Let $R : \mathcal{E}$ and $X : \mathcal{B}$. We say that R *refines* X , or $R \sqsubset X$, if

$$X = p(R)$$



Definition (..) => ...: Refinement

we say that the following diagram commutes

$$\begin{array}{ccc} S & \xrightarrow{\alpha} & R \\ p \downarrow & & \downarrow p \\ X & \xrightarrow{f} & Y \end{array}$$

when $p(\alpha) = f$.

Definition 2 (Cartesian morphism)

Let $R, S : \mathcal{E}$. A morphism $\alpha : S \rightarrow R$ is said to be *cartesian* if, for any $S' : \mathcal{E}$, $\alpha' : S' \rightarrow R$ and $f : p(S') \rightarrow p(S)$ such that the following diagram commutes

$$\begin{array}{ccc} p(S') & \xrightarrow{p(\alpha')} & p(R) \\ f \downarrow & \nearrow p(\alpha) & \\ p(S) & & \end{array}$$

there exists a unique $\hat{f} : S' \rightarrow S$ such that $f = p(\hat{f})$, and such that the following diagram commutes

$$\begin{array}{ccc} S' & \xrightarrow{\alpha'} & R \\ \hat{f} \downarrow & \nearrow \alpha & \\ S & & \end{array}$$



Definition (..) => ...: Cartesian morphism

A (Grothendieck) fibration states that, for every “pullback situation”, there exists a cartesian morphism that “pulls back”.

Definition 3 (Fibration)

p is said to be a *fibration* if, for any

$$\begin{array}{ccc} & & R \\ & & \downarrow p \\ X & \xrightarrow{f} & Y \end{array}$$

there exists a cartesian morphism α making the following diagram commute

$$\begin{array}{ccc} S & \xrightarrow{\alpha} & R \\ \downarrow p & & \downarrow p \\ X & \xrightarrow{f} & Y \end{array}$$



Definition (..) $\Rightarrow$...: Fibration

Fibrations over a given category can be organized into a category.

Definition 4 (Category of fibrations)

For a base category $\mathcal{B}$, define $\mathbf{Fib}_{\mathcal{B}}$ as the category of fibrations over $\mathcal{B}$, that is, whose objects are pairs $(\mathcal{E}, p)$ with $\mathcal{E}$ a category and $p : \mathcal{E} \rightarrow \mathcal{B}$ a fibration.

Given two fibrations $p_i : \mathcal{E}_i \rightarrow \mathcal{B}$ ($i = 1, 2$) over $\mathcal{B}$, a *morphism of fibrations* from p_1 to p_2 is a functor $F : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ making the following diagram commute

$$\begin{array}{ccc} \mathcal{E}_1 & \xrightarrow{F} & \mathcal{E}_2 \\ \downarrow p_1 & & \downarrow p_2 \\ \mathcal{B} & \xrightarrow{\text{id}_{\mathcal{B}}} & \mathcal{B} \end{array}$$

and which preserves cartesian morphisms.



Definition (..) $\Rightarrow$...: Category of fibrations